Convergence of Scalar-Tensor theories toward General Relativity and Primordial Nucleosynthesis

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Abstract. In this paper, we analyze the conditions for convergence toward General Relativity of scalar-tensor gravity theories defined by an arbitrary coupling function α (in the Einstein frame). We show that, in general, the evolution of the scalar field (φ) is governed by two opposite mechanisms: an attraction mechanism which tends to drive scalar-tensor models toward Einstein's theory, and a repulsion mechanism which has the contrary effect. The attraction mechanism dominates the recent epochs of the universe evolution if, and only if, the scalar field and its derivative satisfy certain boundary conditions. Since these conditions for convergence toward general relativity depend on the particular scalar-tensor theory used to describe the universe evolution, the nucleosynthesis bounds on the present value of the coupling function, α_0, strongly differ from some theories to others. For example, in theories defined by $\alpha \propto |\varphi|$ analytical estimates lead to very stringent nucleosynthesis bounds on α_0 ($\lesssim 10^{-19}$). By contrast, in scalar-tensor theories defined by $\alpha \propto \varphi$ much larger limits on α_0 ($\lesssim 10^{-7}$) are found.

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1. Introduction

Scalar-tensor (ST) gravity theories [1, 2, 3] have become a focal point of interest in many areas of gravitational physics and cosmology. They provide the most natural generalizations of General Relativity (GR) by introducing an additional scalar field, ϕ, the dynamical importance of which is determined by an arbitrary coupling function $\omega(\phi)$. Indeed, most recent attempts at unified models of fundamental interactions, i.e., string theories [4] predict the existence of scalar partners to the tensor gravity of GR, and would have ST gravity as their low energy limit. In addition, ST theories are important in cosmology because they provide a natural (non-fine tuned) way of exiting the inflationary epoch.

Solar system experiments (time delay in the Viking Mars data, Lunar Laser Ranging, etc.) can put limits on the present deviation of ST theories with respect to GR. It is admitted [5] that such experiments impose the constraint $\omega_0 > 500$ to the present value of the coupling function and, therefore, that ST theories are at present very close to General Relativity. This limit on ω_0 does not necessarily imply that the universe evolution is, at any time, very close to that found in GR. It has been shown [6, 7] that, in some ST theories, the cosmological evolution drives the scalar field toward a state indistinguishable from GR. Within this 'attraction mechanism', the scalar field can play an important role only in early cosmology because, afterwards, it evolves toward a state with a vanishingly small scalar contribution.

The dynamical importance of the scalar field in the early universe can be checked by means of the observed abundance of light elements, which has to be explained as an outcome of the primordial nucleosynthesis process (PNP). Contrary to the weak field limit, there is not a commonly accepted PNP constraint on ω_0. Some authors [8, 9] have recently found that, in the framework of some ST theories, $\omega_0 \gtrsim 10^7$ is required to obtain the observed primordial abundance of light elements. Other authors [10, 11] have instead found that the PNP test typically imposes the limit $\omega_0 \gtrsim 10^{20}$.

In this paper we will first reexamine the attraction mechanism of Refs. [6, 7] and, then, we will investigate the reason for the enormous discrepancy (thirteenth orders of magnitude) in the PNP bound on ω_0 obtained by different authors. We will show that, in general, the evolution of the scalar field is governed by two opposite mechanisms: an attraction and a repulsion mechanism. The attraction mechanism dominates the recent epochs of the universe evolution only if the scalar field and its derivative satisfy certain boundary conditions which depend on the particular scalar-tensor theory used to describe the universe evolution. We will also apply this generalized formalism to the theories considered in Ref. [11], where the coupling function was restricted to be a monotonic function of time. We will show that the nucleosynthesis bounds numerically obtained in [11] ($\omega_0 \gtrsim 10^{20}$) are in close agreement with the analytical estimates for these theories. When the same arguments are applied to other ST theories (as those of Refs. [6, 7]), one obtains much less stringent bounds on ω_0. Consequently, the particular ST theory used to describe the universe evolution has a crucial importance on the PNP

limit on ω_0 and, therefore, it is not possible to establish a general and unique limit for all ST models.

The plan of the paper is as follows. We begin outlining the scalar-tensor theories as well as the two frames usually considered to build up cosmological models in their framework (Sec. 2). An autonomous evolution equation for the scalar field is then obtained in Sec. 3 both for the Jordan and the Einstein frame. Using this equation, and the particular family of scalar-tensor theories specified in Sec. 4, we then analyze the evolution of the scalar field both in the radiation-dominated epoch (Sect. 5) and the matter-dominated epoch (Sect. 6). Our results are then applied to estimate the nucleosynthesis bounds on ω_0 for this family of theories (Sec. 7). Finally, conclusions and a summary of our results are given in Sec. 8.

2. Scalar-Tensor Gravity Theories

2.1. The Jordan frame

We consider a scalar-tensor gravity theory in which the gravitational interaction is carried by the metric $g_{\mu\nu}$ and an additional massless scalar field, ϕ. Since these fields are measured through laboratory clocks and rods, which are made of atoms and are essentially based on a non-gravitational physics, the action describing a scalar-tensor gravity theory must keep unaltered the basic laws of non-gravitational interactions (as, e.g., statistical physics and electrodynamics). When this fact is taken into account, the resulting space-time units are termed as "atomic", "Jordan" or "physical" units [12, 13, 14, 15].

Using the Jordan frame, the most general action describing a massless scalar-tensor theory of gravitation is [1, 2, 3]

$$S = \frac{1}{16\pi}\int (\phi\mathcal{R} - \frac{\omega(\phi)}{\phi}\phi_{,\mu}\phi^{,\mu})\sqrt{-g}d^4x + S_M \tag{1}$$

where $\mathcal{R}$ is the curvature scalar of the metric $g_{\mu\nu}$, $g \equiv \det(g_{\mu\nu})$, ϕ is the scalar field, and $\omega(\phi)$ is an arbitrary coupling function.

The variation of Eq. (1) with respect to $g_{\mu\nu}$ and ϕ leads to the field equations:

$$\begin{aligned}\mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = & -\frac{8\pi}{\phi}T_{\mu\nu} - \frac{\omega}{\phi^2}(\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi_{,\alpha}\phi^{,\alpha}) \\ & - \frac{1}{\phi}(\phi_{,\mu;\nu} - g_{\mu\nu}\Box\phi) \end{aligned} \tag{2a}$$

$$(3+2\omega)\Box\phi \quad = 8\pi T - \frac{d\omega}{d\phi}\phi_{,\alpha}\phi^{,\alpha} \tag{2b}$$

which satisfy the usual conservation law

$$T^{\mu\nu}_{;\nu} = 0 \tag{3}$$

where $T^{\mu\nu}$ is the energy-momentum tensor and $\Box\phi \equiv g^{\mu\nu}\phi_{,\mu;\nu}$.

For a homogeneous and isotropic universe, the line-element has a Robertson-Walker form and the energy-momentum tensor corresponds to that of a perfect fluid. The field equations $(2a)$ and $(2b)$ then become

$$\frac{8\pi}{3\Phi}\rho = \frac{c^2K}{R^2} + \frac{\dot{R}^2}{R^2} - \frac{\omega}{6}\frac{\dot{\Phi}^2}{\Phi^2} + \frac{\dot{R}\dot{\Phi}}{R\Phi} \tag{4a}$$

$$-\frac{8\pi G}{3\Phi}(\rho + 3P/c^2) = 2\frac{\ddot{R}}{R} + \frac{\dot{R}}{R}\frac{\dot{\Phi}}{\Phi} + \frac{\ddot{\Phi}}{\Phi} + \frac{2\omega}{3}\frac{\dot{\Phi}^2}{\Phi^2} \tag{4b}$$

$$\ddot{\Phi} + 3\frac{\dot{R}}{R}\dot{\Phi} = \frac{1}{(3+2\omega)}[8\pi G(\rho - 3P/c^2) - \frac{d\omega}{d\Phi}\dot{\Phi}^2] \tag{4c}$$

where $K = 0, \pm 1$, $\Phi \equiv G\phi$, $R(t)$ is the scale factor, ρ and P are the energy-mass density and pressure, respectively, and dots mean time derivatives. In addition, we have the usual conservation equation:

$$d(\rho R^3) + (P/c^2)dR^3 = 0 \tag{5}$$

which ensures that the standard laws of non-gravitational interactions are not modified by the presence of a scalar field.

2.2. The Einstein frame

When the metric is assumed to be measured through purely gravitational clocks and rods, the space-time units are termed as "Einstein" or "spin" units. In this frame, the general action describing a massless scalar-tensor theory can be obtained from Eq. (1) by a conformal transformation

$$g_{\mu\nu} = A^2(\varphi)g^*_{\mu\nu} \tag{6a}$$

$$A^2(\varphi) = (\Phi)^{-1} \tag{6b}$$

where $A(\varphi)$ is an arbitrary function related to $\omega(\Phi)$ by

$$\alpha = (3+2\omega)^{-1/2} = \frac{d\ln A}{d\varphi} \tag{7}$$

Using Eqs. (1), $(6a)$, $(6b)$ and (7), one obtains

$$S_* = \frac{c^4}{16\pi G_*}\int(\mathcal{R}_* - 2\varphi_{,\mu}\varphi^{,\mu})\sqrt{-g_*}\frac{d^4x}{c} + S^*_M \tag{8}$$

where G_* is Newton's constant and asterisks denote quantities expressed in Einstein units. Since our measures are based on non-purely gravitational rods and clocks, quantities written in the Einstein frame are not observable. Comparison between theory and observations must be then performed by using the Jordan frame.

From the action (8), the Einstein field equations are:

$$\mathcal{R}^*_{\mu\nu} = 2\varphi_{,\mu}\varphi_{,\nu} + 8\pi G_*(T^*_{\mu\nu} - \frac{1}{2}T^*g^*_{\mu\nu}) \tag{9a}$$

$$\Box_*\varphi = -4\pi G_*\alpha(\varphi)T_* \tag{9b}$$

where $T^*_{\mu\nu} = 2g_*^{-1/2}\delta S_m/\delta g^*_{\mu\nu}$ is the energy-momentum tensor in Einstein units.

If we consider a homogeneous and isotropic universe, the field equations (9a)-(9b) become

$$-3\frac{1}{R_*}\frac{d^2R_*}{dt_*^2} = 4\pi G_*(\rho_* + 3P_*) + 2\left(\frac{d\varphi}{dt_*}\right)^2 \tag{10a}$$

$$3\frac{1}{R_*^2}\left(\frac{dR_*}{dt_*}\right)^2 + 3\frac{K}{R_*^2} = 8\pi G_*\rho_* + \left(\frac{d\varphi}{dt_*}\right)^2 \tag{10b}$$

$$\frac{d^2\varphi}{dt_*^2} + 3\frac{1}{R_*}\frac{dR_*}{dt_*}\frac{d\varphi}{dt_*} = -4\pi G_*\alpha(\varphi)(\rho_* - 3P_*) \tag{10c}$$

and the 'conservation' equation is modified to

$$d(\rho_* R_*^3) + P_* d(R_*^3) = (\rho_* - 3P_*)R_*^3 da(\varphi) \tag{11}$$

Since the mass-energy is not conserved in Einstein units, the basic laws of non-gravitational physics are modified in this frame (see [16, 17, 18] for a reformulation of nuclear reaction rates and thermodynamics in Einstein units).

3. Decoupled evolution of the Jordan scalar field

In the form given above, the time evolution of the scale factor and the scalar field are coupled both in the Jordan and the Einstein frame. Previous works [6, 7] have found that, by introducing an appropriate change of variables, it is possible to find an evolution equation for the Einstein scalar field which is independent of the cosmic scale factor. We will now show that it is also possible to find such a decoupled evolution equation for the Jordan scalar field.

Let us define the functions

$$\psi \equiv \frac{1}{2}\ln\Phi \tag{12a}$$

$$\gamma \equiv \frac{P/c^2}{\rho} \tag{12b}$$

$$\epsilon \equiv \frac{3c^2K\Phi}{8\pi G\rho R^2} \tag{12c}$$

$$W \equiv (3+2\omega)/3 \tag{12d}$$

The Jordan evolution equations of R and Φ then become:

$$\frac{\ddot{R}}{R} + \frac{\dot{R}}{R}\dot{\psi} + \ddot{\psi} + 2W\dot{\psi}^2 = -\frac{4\pi G\rho}{3\Phi}(1+3\gamma) \tag{13a}$$

$$\frac{8\pi G\rho}{3\Phi}(1-\epsilon) = \left(\frac{\dot{R}}{R} + \dot{\psi}\right)^2 - W\dot{\psi}^2 \tag{13b}$$

$$2\ddot{\psi} + 4\dot{\psi}^2 + 6\frac{\dot{R}}{R}\dot{\psi} = \frac{8\pi G\rho}{3\Phi W}(1-3\gamma) - \frac{1}{W}\frac{dW}{d\psi}\dot{\psi}^2 \tag{13c}$$

In order to obtain a decoupled evolution equation for ψ, we will define a 'time' parameter, p, as:

$$dp = h_c dt; \qquad h_c = \frac{\dot{R}}{R} + \dot{\psi} \tag{14}$$

In terms of these variables, and denoting $f' \equiv df/dp$, Eqs. (13a)-(13c) reduce to:

$$\frac{h_c'}{h_c} = \psi' - 1 - 2W\psi'^2 - \frac{4\pi G\rho}{3\Phi h_c^2}(1+3\gamma) \tag{15}$$

$$h_c^2(1 - W\psi'^2) = \frac{8\pi G\rho}{3\Phi}(1-\epsilon) \tag{16}$$

$$\psi'' + \frac{h_c'}{h_c}\psi' - \psi'^2 + 3\psi' =$$

$$= \frac{4\pi G\rho}{3\Phi h_c^2}\frac{1-3\gamma}{W} - \frac{1}{2W}\frac{dW}{d\psi}\psi'^2 \tag{17}$$

and, using Eqs. (16) and (15) to eliminate h_c and h_c', respectively, we finally obtain:

$$\frac{2(1-\epsilon)}{1-W\psi'^2}\psi'' + (3 - 3\gamma - 4\epsilon)\psi' =$$

$$= \frac{1-3\gamma}{W} - \left(\frac{1-\epsilon}{1-W\psi'^2}\right)\frac{1}{W}\frac{dW}{d\psi}\psi'^2 \tag{18}$$

which is an evolution equation for $\psi = (1/2)\ln\Phi$ independent of the evolution of the cosmic scale factor. This equation is the analogous, in the Jordan frame, to Eq. (3.15) of [6]. Therefore, we find that a decoupled evolution of the scalar field is not an exclusive feature of the Einstein frame.

In order to find the well-known decoupled evolution equation for the Einstein scalar field, we will first multiply Eq. (18) by $W^{1/2}(1 - W\psi'^2)/2(1-\epsilon)$. We obtain:

$$\frac{2(1-\epsilon)}{1-W\psi'^2}(W^{1/2}\psi')' =$$

$$= \frac{1-3\gamma}{W^{1/2}} - 3(1 - \gamma - \frac{4}{3}\epsilon)W^{1/2}\psi' \tag{19}$$

We can now change to the Einstein frame by performing the conformal transformation (6a)-(6b) with

$$\frac{d\varphi}{d\psi} = -\sqrt{3W} = -\alpha^{-1} \tag{20}$$

Eq. (19) then becomes:

$$\frac{2(1-\epsilon)}{3-\varphi'^2}\varphi'' + (1 - \gamma - \frac{4}{3}\epsilon)\varphi' = -\alpha(1-3\gamma) \tag{21}$$

which agrees with Eq. (3.15) of [6].

We also note that, by defining H_* and ρ_* through

$$\Phi^{1/2}H_* = h_c; \qquad \rho = \Phi^2\rho_*, \tag{22}$$

Eq. (16) yields

$$\frac{8\pi G\rho_*}{H_*^2} = \frac{3-\varphi'^2}{1-\epsilon} \tag{23}$$

From this equation we see that, when $\epsilon = 0$, the local positivity of the energy density implies that

$$\varphi'^2 \leq 3. \tag{24}$$

Most papers analyzing the convergence toward General Relativity of scalar-tensor theories are based on the Einstein frame. In order to make easier the comparison of our results with those founds in previous works, we will use hereafter the Einstein frame. Nevertheless, it must be noted that most of our conclusions can also be found by using the Jordan evolution equation (18).

4. The coupling function

The time evolution of scalar-tensor theories can be studied only after specifying a functional form of the coupling function. Barrow and Parsons [19] have noted that most expressions used in the literature for $(3+2\omega)$ can be classified into three different families of theories:

$$\text{a) Theories-1:} \quad 3+2\omega = \frac{1}{B_1|\Phi-1|^{\delta}} \quad (\delta > 1/2) \tag{25a}$$

$$\text{b) Theories-2:} \quad 3+2\omega = \frac{1}{B_1|\ln\Phi|^{\delta}} \quad (\delta > 1/2) \tag{25b}$$

$$\text{c) Theories-3:} \quad 3+2\omega = \frac{1}{B_1|\Phi^{\delta}-1|} \quad (\delta > 0) \tag{25c}$$

where B_1 is an arbitrary positive constant.

The three theories defined by Eqs. (25a)-(25c) imply very different behaviours of the early universe, which have been analyzed in detail by Barrow and Parsons [19]. The first class of theories has also been studied by García-Bellido and Quirós [20], Serna and Alimi [21], Comer et al. [22], and Navarro et al. [23]. However, it is important to note that all these theories have similar behaviours in the limit close to General Relativity $(\Phi \to 1)$. As a matter of fact, since $\ln\Phi \simeq \Phi - 1$ and $\Phi^{\delta} - 1 \simeq \delta \ln\Phi$, we can take

$$3+2\omega = \frac{1}{B_1|\ln\Phi|^{\delta}} = \frac{1}{B_1|2\psi|^{\delta}} \quad (\delta > 1/2) \tag{26}$$

to represent the way in which these three types of theories approach the limit of GR.

Using the coupling function given by Eq. (26), integration of Eq. (20) yields

$$\varphi = -\frac{\mathrm{sign}(\psi)}{(2-\delta)B_1^{1/2}}|2\psi|^{(2-\delta)/2} \quad (\delta < 2) \tag{27}$$

where we have normalized the integration constant so that $\varphi = 0$ corresponds to $\Phi = 1$ (or $\psi = 0$). Note that, according to Eq. (27), $\mathrm{sign}(\varphi) = -\mathrm{sign}(\psi)$.

By introducing Eq. (27) into Eq. (26), we obtain the Einstein form of the coupling function:

$$\alpha = B_2|\varphi|^{\frac{\delta}{2-\delta}} = \kappa(\varphi)|\varphi| \tag{28}$$

where $B_2 \equiv B_1^{1/(2-\delta)}(2-\delta)^{\delta/(2-\delta)} > 0$, and

$$\kappa(\varphi) = B_2|\varphi|^{\frac{2(\delta-1)}{2-\delta}} \tag{29}$$

5. Radiation-dominated evolution

In the radiation-dominated epoch, the state equation is $P/c^2 = \rho/3$ (i.e., $\gamma = 1/3$) and the curvature effects are negligible ($\epsilon = 0$). Consequently, the evolution equation of the Einstein scalar field (Eq. 21) is well approximated by:

$$\frac{2}{3-\varphi'^2}\varphi'' + \frac{2}{3}\varphi' = 0 \tag{30}$$

which does not depend on the functional form of $\alpha(\varphi)$.

The integration of Eq. (30) gives:

$$\varphi'^2 = \frac{3k^2}{e^{2p}+k^2} \tag{31}$$

where k is related to the initial ($p = 0$) velocity φ'_R through:

$$k^2 = \frac{(\varphi'_R)^2}{3-(\varphi'_R)^2} \tag{32}$$

In terms of k, the solution of Eq. (30) is:

$$\varphi = \varphi_R - \sqrt{3}\,\mathrm{sign}(k)\ln\left[\frac{\sqrt{1+k^2e^{-2p}}+ke^{-p}}{\sqrt{1+k^2}+k}\right] \tag{33}$$

6. Matter-dominated evolution: The attraction-repulsion mechanism

Let us now analyze the evolution of the scalar field during the matter-dominated era ($\gamma = 0$) of a flat universe ($\epsilon = 0$). The evolution equation for the Einstein scalar field is, in this case:

$$\frac{2}{3-\varphi'^2}\varphi'' + \varphi' + \alpha = 0 \tag{34}$$

As in previous works [6, 8], we will first assume that, at some time (for instance, at the beginning of the matter-dominated era), the scalar-tensor theory is not very far from GR so that we can neglect φ'^2 against 3 and, in addition, the coupling function $\alpha(\varphi)$ is represented by Eq. (28).

In this case, the evolution equation (34) reduces to:

$$\frac{2}{3}\varphi'' + \varphi' + \sigma_\varphi\kappa(\varphi)\varphi = 0 \tag{35}$$

where

$$\sigma_\varphi = \mathrm{sign}(\varphi) \tag{36}$$

When $\sigma_\varphi = +1$, the above expression corresponds to the evolution equation of a damped harmonic oscillator with a variable elastic coefficient $\kappa(\varphi)$. The first term ($2\varphi''/3$) represents the total force on a fictitious particle of mass $m = 2/3$. The second term (φ') corresponds to a friction force proportional to the velocity and, finally, the third term represents an elastic force with a variable coefficient $\kappa(\varphi)$. The existence, in these conditions, of a damped oscillatory behaviour of the Einstein scalar field was

first reported by Damour and Nordvedt [6] and it is usually termed as the 'attraction mechanism' toward General Relativity .

We note however that, if $\sigma_\varphi = -1$, the effective elastic coefficient $\sigma_\varphi \kappa(\varphi)$ is negative and, consequently, there exists a 'repulsion' mechanism instead an attraction one. We will now analyze the matter-dominated evolution of the scalar field by considering separately the cases $\delta = 1$ and $\delta \neq 1$ in Eq. (29).

It must be noted that the class of scalar-tensor theories analyzed in [6] correspond to a positive constant σ_φ and, therefore, they are always attractive.

6.1. Case of a constant elastic coefficient ($\delta = 1$)

When $\delta = 1$, the elastic coefficient defined by Eq. (29) becomes a positive constant B_2. The scalar field evolution equation then reduces to:

$$\varphi'' + \frac{3}{2}\varphi' + \frac{3}{2}\sigma_\varphi B_2 \varphi = 0 \tag{37}$$

Since the roots of the characteristic equation are:

$$r_\pm = -\frac{3}{4} \pm \frac{3}{4}\sqrt{1 - \frac{8}{3}B_2 \sigma_\varphi} \tag{38}$$

the general solution of Eq. (37) admits four different behaviours, depending on B_2 and σ_φ:

a) Damped harmonic motion ($\sigma_\varphi > 0$, $B_2 > 3/8$)

$$\varphi = C_1 e^{-\frac{3}{4}p} \cos(\omega_1 p + C_2) \tag{39}$$

where

$$\omega_1 = \frac{3}{4}\sqrt{\frac{8}{3}B_2 - 1} \tag{40a}$$

$$C_1 = \varphi_0 \left[\left(\frac{\varphi_0' + \frac{3}{4}\varphi_0}{\omega_1 \varphi_0} \right)^2 + 1 \right]^{1/2} \tag{40b}$$

$$C_2 = -\mathrm{atan} \left(\frac{\varphi_0' + \frac{3}{4}\varphi_0}{\omega_1 \varphi_0} \right) \tag{40c}$$

b) Critically damped motion ($\sigma_\varphi > 0$, $B_2 = 3/8$):

$$\varphi = e^{-\frac{3}{4}p}(C_1 p + C_2) \tag{41}$$

where

$$C_1 = \varphi_0' + \frac{3}{4}\varphi_0; \quad C_2 = \varphi_0 \tag{42}$$

c) Overdamped attraction motion ($\sigma_\varphi > 0$, $B_2 < 3/8$):

$$\varphi = e^{-\frac{3}{4}p}[C_1 e^{\beta_1 p} + C_2 e^{-\beta_1 p}] \tag{43}$$

where

$$\beta_1 = \frac{3}{4}\sqrt{1 - \frac{8}{3}B_2} < \frac{3}{4} \tag{44a}$$

$$C_1 = \frac{\varphi_0' + (\beta_1 + 3/4)\varphi_0}{2\beta_1}; \tag{44b}$$

$$C_2 = -\frac{\varphi_0' + (-\beta_1 + 3/4)\varphi_0}{2\beta_1} \tag{44c}$$

d) Overdamped repulsion motion ($\sigma_\varphi < 0$, and any B_2):

$$\varphi = e^{-\frac{3}{4}p}[C_1 e^{\beta_2 p} + C_2 e^{-\beta_2 p}] \tag{45}$$

where

$$\beta_2 = \frac{3}{4}\sqrt{1 + \frac{8}{3}B_2} > \frac{3}{4} \tag{46a}$$

$$C_1 = \frac{\varphi_0' + (3/4 + \beta_2)\varphi_0}{2\beta_2}; \tag{46b}$$

$$C_2 = -\frac{\varphi_0' + (3/4 - \beta_2)\varphi_0}{2\beta_2} \tag{46c}$$

Apparently, only the first three solutions converge toward GR ($\varphi = 0$) while the last one diverges to $\pm\infty$ as $p \to \infty$. However, it must be noted that each of such solutions governs the whole matter-dominated era if, and only if, σ_φ does not change. This is the case of the models analyzed in Refs. [6, 7] where, in Eq. (37), σ_φ is forced to be $\sigma_\varphi \equiv +1$ at any time.

In the case of the models considered here, where the sign σ_φ of φ can be variable in time, the matter-dominated evolution of the scalar field has a much more complicated behaviour. As a matter of fact, the above four solutions (Eqs. 39–45) admit the possibility of a change in the sign of φ at a finite time $p > 0$ (this is specially obvious for the first solution due to the oscillatory behaviour of the cosinus function). Therefore, an initially convergent model ($\sigma_\varphi = +1$) could finally diverge from GR as a consequence of a change in the sign of φ. Reciprocally, an initially divergent model ($\sigma_\varphi = -1$) could finally converge toward GR if sign(φ) changes. We must then perform a more detailed analysis to find the conditions for convergence (or divergence) with respect to GR.

According to Eqs. (39)–(46c), φ vanishes (and, therefore, σ_φ changes) at a finite $p > 0$ if, and only if, the initial values of φ and φ' satisfy the following conditions:

$$\begin{cases} \text{a)} & \text{always} \\ \text{b)} & \varphi_0' < -\frac{3}{4}\varphi_0, \qquad (\varphi_0 > 0) \\ \text{c)} & \varphi_0' < -(\frac{3}{4} + \beta_1)\varphi_0, \qquad (\varphi_0 > 0) \\ \text{d)} & \varphi_0' > -(\frac{3}{4} + \beta_2)\varphi_0, \qquad (\varphi_0 < 0) \end{cases} \tag{47}$$

When these conditions are satisfied, an initially attraction behaviour becomes a repulsion one at $p_1 > 0$. Reciprocally, an initially repulsion behaviour becomes, at $p_1 > 0$, an attraction one. The question is now to determine whether these new behaviours, reached at $p > p_1$, govern the rest of the matter-dominated evolution.

Let us first consider the attraction-to-repulsion case. In this situation, the attraction models given by Eqs. (39), (41) and (43) (with $p \leq p_1$) imply a vanishing scalar field at $p = p_1$. After this time, the sign of φ has changed and, therefore, all models become expressed by Eq. (45) but with the integration constants (see Eqs. 46b and 46c) given by:

$$C_1 = -C_2 = \frac{\varphi_1'}{2\beta_2} \tag{48}$$

where φ_1' denotes the φ' value at $p = p_1$, and we have taken into account that $\varphi(p_1) = 0$.

Using Eqs. (45) and (48), the scalar field evolution for $p \geq p_1$ is given by:

$$\varphi = \frac{\varphi_1'}{2\beta_2} e^{-\frac{3}{4}\tilde{p}}[e^{\beta_2\tilde{p}} - e^{-\beta_2\tilde{p}}] \tag{49}$$

where $\tilde{p} \equiv p - p_1$.

This solution never vanishes for $p > p_1$ and diverges to $\pm\infty$ when $p \to +\infty$. Consequently, none of these models finally converges towards GR.

Let us now consider the repulsion-to-attraction case. In this situation, the repulsion model given by Eq. (45) (with $p \leq p_1$) implies a vanishing scalar field at $p = p_1$. After this time, the sign of φ changes and, therefore, the scalar field evolution becomes expressed by Eqs. (39), (41) or (43), depending on the B_2 value. The integration constants in this new regime are:

$$\begin{cases} C_1 = \frac{|\varphi_1'|}{\omega_1}, & C_2 = \pi/2 \quad (B_2 > 3/8) \\ C_1 = \varphi_1', & C_2 = 0 \qquad\ \ (B_2 = 3/8) \\ C_1 = \frac{\varphi_1'}{2\beta_1}, & C_2 = -C_1 \quad (B_2 < 3/8) \end{cases} \tag{50}$$

Therefore, when $p \geq p_1$:

$$\varphi = \begin{cases} \frac{|\varphi_1'|}{\omega_1} e^{-\frac{3}{4}\tilde{p}} \sin(\omega_1\tilde{p}) & (B_2 > 3/8) \\ \varphi_1' e^{-\frac{3}{4}\tilde{p}}\tilde{p} & (B_2 = 3/8) \\ \frac{\varphi_1'}{2\beta_1} e^{-\frac{3}{4}\tilde{p}}[e^{\beta_1\tilde{p}} - e^{-\beta_1\tilde{p}}] & (B_2 < 3/8) \end{cases} \tag{51}$$

We then find that, if $B_2 > 3/8$, the new attraction behaviour is oscillatory. Consequently, it will return later to have a repulsion motion similar to that of Eq. (49), and these models will finally diverge from GR. On the contrary, if $B_2 \leq 3/8$, Eqs. (51) imply that φ never vanishes at $p > p_1$ except for $p \to +\infty$ and, therefore, these models are finally attracted toward GR.

Summarizing, all models with $B_2 > 3/8$ diverge from GR, whatever the initial values of σ_φ, φ and φ' are. In the same way, models with $B_2 \leq 3/8$ and $\sigma_\varphi > 0$ (at $p = 0$) will diverge from GR if conditions (47)b or (47)c are satisfied, while models with $\sigma_\varphi < 0$ (at $p = 0$) will diverge from GR if conditions (47)d are not satisfied. Consequently (see Fig. 1), the scalar-tensor theories defined by Eq. (28) with $\delta = 1$ will converge toward GR if, and only if $B_2 \leq 3/8$ and:

$$\varphi_0' \geq -(3/4 + \beta_1)\varphi_0 \quad \text{(for any } \sigma_\varphi \text{ value)} \tag{52}$$

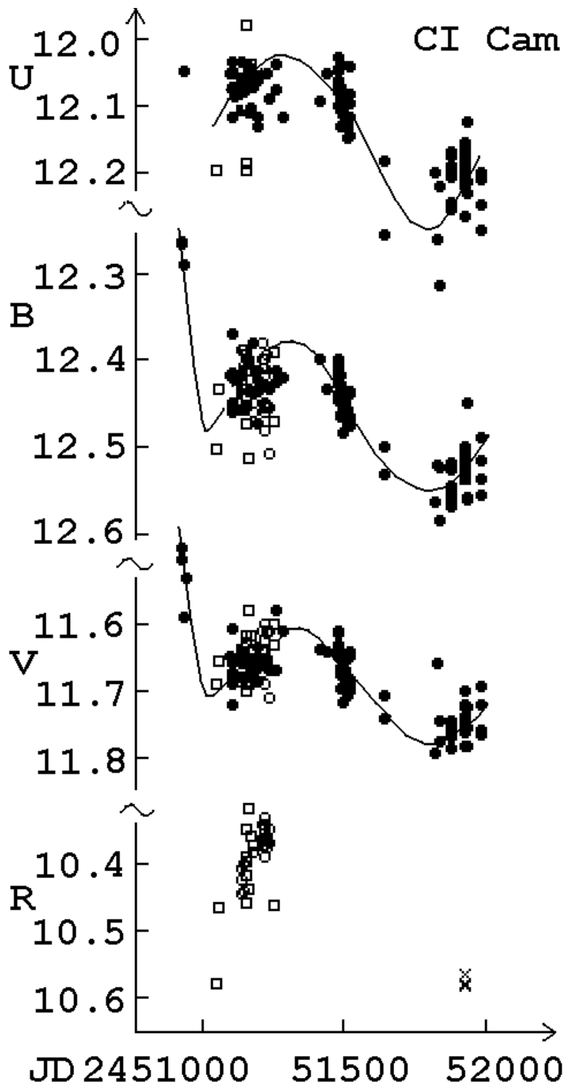

Figure 1. Matter-dominated evolution of φ in converging models with $\varphi_0 < 0$ and $\varphi_0' = \varphi_{crit}$ (solid line), $\varphi_0 < 0$ and $\varphi_0' > \varphi_{crit}$ (dotted line), $\varphi_0 > 0$ and $\varphi_0' = \varphi_{crit}$ (dashed line), and $\varphi_0 > 0$ and $\varphi_0' > \varphi_{crit}$ (dotted-dashed line), where $\varphi_{crit} \equiv -(3/4 + \beta)\varphi_0$.

This result can be understood by considering the scalar field φ as the position of a particlelike dynamical variable which moves under a potential $V(\varphi)$. Since the right-hand side of Eq. (21) represents a force term, this potential is given by:

$$V(\varphi) = (1 - 3\gamma) \int \alpha(\varphi) d\varphi \tag{53}$$

If $\alpha(\varphi)$ is given by Eq. (28) with $\delta = 1$, the matter-dominated form of $V(\varphi)$ becomes:

$$V(\varphi) = \frac{1}{2}\sigma_\varphi B_2 \varphi^2 \tag{54}$$

where we have taken the origin of potentials so that $V = 0$ at $\varphi = 0$.

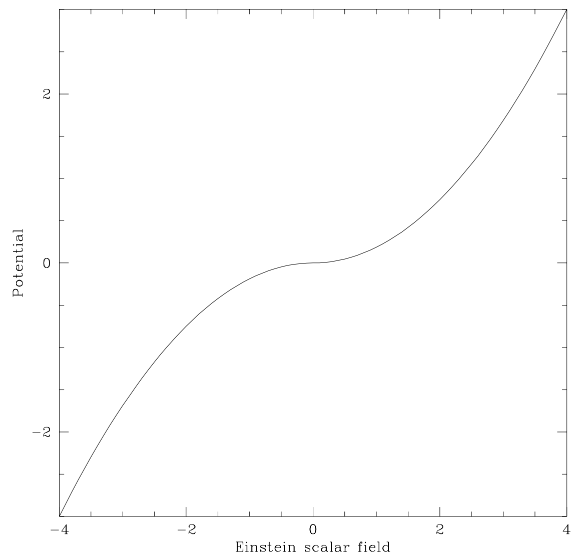

Figure 2. Potential $V(\varphi)$ for scalar-tensor theories with $\delta = 1$. Here, the particular case $B_2 = 1/9$ is shown

Figure 2 shows this potential. We see that $V(\varphi)$ has a stationary point at $\varphi = 0$ which does not correspond to a local minimum or maximum. Particles with $\varphi > 0$ are attracted towards $\varphi = 0$ while those with $\varphi < 0$ are rejected from $\varphi = 0$.

In the case of ST theories characterized by an inefficient friction ($B_2 > 3/8$), any particle will 'slide' on this potential and will move toward $\varphi \to -\infty$, whatever the initial values of φ and φ' are. This explains our previous result that any theory with $B_2 > 3/8$ diverges from GR.

In the opposite case of theories characterized by an efficient enough friction ($B_2 \leq 3/8$), particles with $\varphi > 0$ have an overdamped or critically damped motion. Therefore, if the initial velocity φ'_0 is not more negative than a critical value $-(\beta_1 + 3/4)\varphi_0$, the energy is completely dissipated by the friction term, and the particle stops at $\varphi = 0$. On the contrary, if $\varphi < 0$, the tendency to diverge from GR ($\varphi = 0$) can only be overcome if the particle has an initial velocity high enough to reach a point placed in the attraction ($\varphi > 0$) region. This explains our previous result that, in ST theories with $B_2 \leq 3/8$, the convergence towards GR impose on φ'_0 the constraints given by Eqs. (52).

6.2. Case of a variable elastic coefficient ($\delta \neq 1$)

When $\delta \neq 1$, the matter-dominated potential is given by

$$V(\varphi) = \frac{2-\delta}{2}\sigma_\varphi B_2 |\varphi|^{2/(2-\delta)} \tag{55}$$

Since the form of this potential is qualitatively similar to that of Eq. (54), we can expect for these models the same types of behaviours as those found in the $\delta = 1$ case.

Nevertheless, as we have seen in the previous subsection, these behaviours strongly depend on the relative importance of the friction term and on the initial value of φ and φ'. We can obtain some insight on the conditions for convergence toward GR by considering the two following limiting cases.

When $1 < \delta < 2$, the elastic coefficient defined by Eq. (29) tends to zero as $\varphi \to 0$. Consequently, in the limit close enough to GR, we can neglect the term $\sigma_\varphi \kappa(\varphi)\varphi$ of Eq.(35), so that the scalar field evolution equation becomes *friction dominated*:

$$\varphi'' + \frac{3}{2}\varphi' = 0 \tag{56}$$

The solution of this equation is:

$$\varphi = C_1 + C_2 e^{-\frac{3}{2}p} = \varphi_0 + \frac{2}{3}\varphi_0'(1 - e^{-\frac{3}{2}p}) \tag{57}$$

which implies that φ vanishes at a finite time $p_1 > 0$ provided that:

$$\varphi_0' > -\frac{3}{2}\varphi_0 \quad (\varphi_0 < 0) \tag{58a}$$

$$\varphi_0' < -\frac{3}{2}\varphi_0 \quad (\varphi_0 > 0) \tag{58b}$$

These expressions correspond to the limit $B_2 \to 0$ (very strong friction) of Eq. (47) for the $\delta = 1$ case. Therefore, according to Eqs. (52), theories with $\delta > 1$ will converge toward GR provided that

$$\varphi_0' \geq -\frac{3}{2}\varphi_0 \tag{59}$$

On the contrary, when $1/2 < \delta < 1$, the elastic coefficient becomes infinity as $\varphi \to 0$ (see Eq. 29). The friction term of Eq. (21) is then very inefficient and, therefore, ST theories behave as the limit $B_2 \to \infty$ (very weak friction) of the $\delta = 1$ case. That is, the φ particle will 'slide' on the potential and will move towards $\pm\infty$. These models will then diverge from GR whatever the initial values of φ and φ' are.

6.3. Influence of the mass term

In the preceding discussion, we have obtained necessary and sufficient conditions for convergence toward GR in the limit of small φ' velocities. We will now show that, in the most general situation with a non-negligible φ', there also exists a necessary (but not sufficient) condition for convergence toward GR.

Let us consider Eq. (34) with the coupling function defined in Eq. (28):

$$\frac{2}{3 - \varphi'^2}\varphi'' + \varphi' = -\sigma_\varphi \kappa(\varphi)\varphi \tag{60}$$

By integrating this equation, one obtains

$$\begin{aligned} \ln\left(\frac{\sqrt{3}+\varphi'}{\sqrt{3}+\varphi_0'}\,\frac{\sqrt{3}-\varphi_0'}{\sqrt{3}-\varphi'}\right)^{\frac{1}{\sqrt{3}}} &+ (\varphi - \varphi_0) = \\ &= -\,B_2\int_0^p |\varphi|^{\frac{\delta}{2-\delta}}\,dp \end{aligned} \tag{61}$$

Let us now assume that theories converge toward GR as $p \to +\infty$. In this case, $\varphi \to 0$ and $\varphi' \to 0$ as $p \to +\infty$, so that Eq. (61) becomes

$$\ln\left(\frac{\sqrt{3}+\varphi_0'}{\sqrt{3}-\varphi_0'}\right)^{\frac{1}{\sqrt{3}}} + \varphi_0 = B_2 \int_0^{\infty} |\varphi|^{\frac{\delta}{2-\delta}} dp > 0 \tag{62}$$

Therefore, the assumption of convergence toward GR lead to the necessary condition:

$$\varphi_0' > \sqrt{3}\frac{1-e^{\sqrt{3}\varphi_0}}{1+e^{\sqrt{3}\varphi_0}} \tag{63}$$

which leads to $\varphi_0' > -3\varphi_0/2$ when $\varphi_0 \ll 1$ while, in the limits $\varphi_0 \to \pm\infty$, implies $\varphi_0' > \mp 1$

7. Implications on Primordial Nucleosynthesis

The primordial nucleosynthesis process (PNP) starts, in the early universe, soon after the cosmic temperature becomes lower than that needed to maintain the proton-to-neutron ratio in its equilibrium value (freezing-out temperature). The nuclear reactions that then take place lead first to the Deuterium and Helium-3 formation. These elements are then burnt to produce Helium-4 and very small amounts of heavier elements. This process is sensitive to the universe expansion rate during nucleosynthesis, which can be parametrized by the speed-up factor $\xi \equiv H/H_{GR}$, where H is the physical Hubble parameter and H_{GR} is that predicted by GR at the same temperature. Since scalar-tensor theories predict a universe expansion rate which differs from that obtained in the framework of GR, the light-element production can be used to place constraints on the ξ value at the beginning of the PNP process, ξ_{PNP}.

Most of works agree on the PNP constraint on ξ_{PNP} (subscripts PNP denote values at the beginning of nucleosynthesis). When a conservative choice for the observed primordial abundances is considered, ξ_{PNP} is constrained to be $0.8 \lesssim \xi_{PNP} \lesssim 1.2$ [9, 24]. However, when a more severe choice for the observed abundances is taken, the PNP limits on ξ_{PNP} are $0.95 \lesssim \xi_{PNP} \lesssim 1.03$ [11, 25]. The problem arises when this bound is used to place constraints on the present value of the coupling function, ω_0.

Serna & Alimi [11] have considered a family of scalar-tensor theories where the matter-dominated potential $V(\varphi)$ is similar to that of Fig. 2 and, in addition, the coupling function $3+2\omega$ is *a monotonic function of time* (i.e., $3+2\omega$ diverges to infinity only in the limit of very large times). Except for some particular cases [26], the nucleosynthesis bounds obtained on these theories are very stringent: $\omega_0 \gtrsim 10^{20}$. Other authors [8, 9] have instead considered a different family of theories where $V(\varphi)$ has a constant sign and $3+2\omega$ is not necessarily a monotonic function of time. The PNP bounds on these last theories are about thirteenth orders of magnitude lower ($\omega_0 \gtrsim 10^7$) than those obtained in the previous case.

It has been suggested in the literature that the above discrepancy in the PNP bounds on ω_0 could be perhaps due to the existence in Ref. [11] of numerical instabilities

(runaway solutions) arising when the field equations are solved from a backward time-integration. We will now show that the integration sense has no effects on the PNP bounds (see the Appendix A for a test on the numerical stability against the time-integration sense). The important point to explain the above discrepancies is the particular scalar-tensor theory used to describe the universe evolution.

We will consider the same family of scalar-tensor theories as in Ref. [11], and we will restrict our discussion to singular models with $3+2\omega > 0$ and a monotonic evolution of the speed-up factor.

Let us first consider the case $\Phi > 1$ ($\varphi < 0$), where $\sigma_\varphi < 0$ and the scalar field enters the matter-dominated era with the overdamped repulsion motion of Eq. (45). According to Eq. (52), the convergence toward GR requires an initial scalar field velocity, φ'_M (subscripts M denote values at the beginning of the matter-dominated era), with a positive value:

$$\varphi'_M \geq -\left[\frac{3}{4} + \beta_2\right] \varphi_M \tag{64}$$

Therefore, since $\varphi_M < 0$, the choice $\varphi'_M = 0$ is *forbidden* for this family of scalar-tensor theories. In addition, the requirement of a monotonic time evolution for the coupling function eliminates the possibility of $\varphi'_M > -(3/4+\beta_2)\varphi_M$, where convergence toward GR is obtained after a rather complicate behaviour of $3+2\omega$ in which it becomes infinity (when $\varphi = 0$ for the first time), then it decreases until reaching a local minimum value (maximum value of φ), and finally it increases monotonically to infinity (see Fig. 2).

Taking the only possible choice for these theories

$$\varphi'_M = -\left[\frac{3}{4} + \beta_2\right] \varphi_M, \tag{65}$$

Eq. (45) reduces to:

$$\varphi = \varphi_M e^{-(\frac{3}{4}+\beta_2)p} \tag{66}$$

where, from Eqs. (46b)-(46c), β_2 has a constant value larger than $3/4$ (very overdamped motion, $B_2 \to 0$) and smaller than $3\sqrt{2}/4$ (critically damped motion, $B_2 \to 3/8$).

Using Eqs. (28) and (66), the present value of the coupling function is (with $\delta = 1$):

$$\alpha_0^2 = B_2^2 \varphi_M^2 e^{-2(\frac{3}{4}+\beta_2)p_0} \tag{67}$$

where $p_0 \approx 10$ (see Ref. [6]) is the amount of p time elapsed since the beginning of the matter-dominated era. Taking the less stringent value $B_2 \approx 1/9$, Eq. (67) implies

$$\alpha_0^2 \approx 10^{-16} \varphi_M^2 \tag{68}$$

while using the largest value $B_2 = 3/8$

$$\alpha_0^2 \approx 10^{-17} \varphi_M^2 \tag{69}$$

In order to estimate α_0^2, we must specify a φ_M value compatible with the observed abundance of light elements. To that end we note that, according to Eqs. (14) and

(22), the physical Hubble parameter H is related to that measured in Einstein units H_* by

$$H^2 = \frac{H_*^2}{A^2}(1 + \alpha\varphi')^2 \tag{70}$$

Therefore, using Eq. (23), we obtain

$$H^2 = \frac{8\pi G\rho A^2}{3 - \varphi'^2}(1 + \alpha\varphi')^2 \tag{71}$$

and the speed-up factor $\xi \equiv H/H_{GR}$ (where $H_{GR}^2 = 8\pi G\rho/3$) is given by:

$$\xi^2 = \frac{3A^2}{3 - \varphi'^2}(1 + \alpha\varphi')^2 \tag{72}$$

which, in the limit of very small φ' values, implies $\xi \simeq A$

In models with a monotonic evolution of the speed-up factor, the compatibility with the observed abundance of light elements imposes $0.95 \lesssim \xi \lesssim 1.03$ just before the big-bang nucleosynthesis process. Since $\xi \simeq A$ and $\psi = -lnA$, this constraint implies (see Eq. 27) :

$$\varphi_{PPN} \approx \begin{cases} -0.72 & (\text{if } B_2 = 1/9) \\ -0.39 & (\text{if } B_2 = 3/8) \end{cases} \tag{73}$$

Taking into account the condition (65) for convergence toward GR (i.e., $\varphi'_M \approx -\varphi_M$ at the end of the radiation era), Eq. (31) implies $k \approx -\varphi_M e^{p_R}/\sqrt{3}$ and Eq. (33) then yields

$$\varphi_{PPN} \approx \varphi_M - \sqrt{3}\ln\left[\frac{\sqrt{\varphi_M^2 e^{2p_R} + 3} - \varphi_M e^{p_R}}{\sqrt{\varphi_M^2 + 3} - \varphi_M}\right] \tag{74}$$

where p_R denotes the p-time elapsed from the relevant epochs prior to the big bang nucleosynthesis up to the end of the radiation era. Using φ_{PPN} values of Eq. (73) and the rather conservative value $p_R = 3$ (i.e., the cosmic temperature decreases in about three orders of magnitude) , Eq. (74) implies:

$$\varphi_M \lesssim \begin{cases} -3.7 \cdot 10^{-2} & (\text{if } B_2 = 1/9) \\ -2.0 \cdot 10^{-2} & (\text{if } B_2 = 3/8) \end{cases} \tag{75}$$

Consequently,

$$\alpha_0^2 \lesssim \begin{cases} 10^{-19} & (\text{if } B_2 = 1/9) \\ 10^{-21} & (\text{if } B_2 = 3/8) \end{cases} \tag{76}$$

in good accord with the numerical computations of Serna and Alimi [11].

The case $\Phi < 1$ ($\varphi > 0$) requires a different analysis. Since the models $3 + 2\omega > 0$ and $\Phi < 1$ studied in [11] are all nonsingular (class-3), the cosmic temperature has a maximum value, T_{max}, which corresponds to the minimum of the scale factor. Consequently, in this class of models, the bounds on α_0 are mainly imposed by the condition that T_{max} must be high enough to allow for the existence itself of primordial nucleosynthesis. The numerical computations of [11] lead to limits on α_0^2 close to those

given by Eq. (76). This nonsingular behaviour is found for any φ'_M value, except for the (fine tuned) choice $\varphi'_M = 0$ where the scalar field evolution is frozen to a constant value during the radiation epoch.

It is worth commenting the fact that the strong limit expressed in Eq. (76) refers to scalar-tensor models with a monotonic evolution of $\xi(T)$. As we remarked in previous works [11], these limits lead to cosmological models which do not significantly differ from the standard GR one except perhaps for very early epochs before nucleosynthesis. On the contrary, when the speed-up factor has not a monotonic behaviour during the big-bang nucleosynthesis, the compatibility with the observed abundance of light elements is possible even for very large ξ_{PPN} values [11, 26]. In the framework of this last family of models, the nucleosynthesis constraints on φ_M and α_0^2 are not very stringent and, in addition, the allowed range for the baryon density is much wider than in the standard GR cosmologies.

8. Conclusions

In this paper we have analyzed the convergence of scalar-tensor theories (ST) toward GR and its consequences on the nucleosynthesis bounds on the present value of the coupling function.

To that end, we have deduced an autonomous evolution equation for the Jordan scalar field. By writing this equation in Einstein units, we have analyzed the evolution of the scalar field both in the radiation-dominated epoch and in the matter-dominated epoch. We have considered a coupling function defined by Eq. (28), which reproduces all the models proposed by Barrow & Parsons [19] in the limit close to GR. We have then shown that, in general, the evolution of the scalar field is governed by two opposite mechanisms: an attraction and a repulsion mechanism. The attraction mechanism dominates the recent epochs of the universe evolution if, and only if, the scalar field and its derivative satisfy certain boundary conditions which depend on each particular scalar-tensor theory.

Our results have been then applied to obtain an analytical estimate of the Big-Bang nucleosynthesis (BBN) bounds on ω_0. We have found that the particular ST theory used to describe the universe evolution has a crucial importance on the BBN limits on ω_0. Therefore, it is not possible to establish a general and unique limit for all ST models. In the particular case of the theories analyzed in this paper, where $\alpha \propto |\varphi|$, our analytical estimates are in close agreement with the nucleosynthesis bounds numerically obtained in [11] ($\alpha_0^2 \lesssim 10^{-20}$). Theories different from those analyzed in this paper could imply very different BBN bounds. For instance, in the case of a ST theory defined by $\alpha(\varphi) \propto \varphi$, where only the attraction mechanism is present, the BBN bounds are ($\alpha_0^2 \lesssim 10^{-7}$) [7, 9]. In the same way, in scalar-tensor theories with a non-monotonic evolution of the speed-up factor, the BNB limits are drastically relaxed to a value comparable to that obtained from solar system experiments ($\alpha_0^2 \lesssim 0.02$). In addition, in this last case, the allowed range for the baryon density is much wider than in the standard GR cosmologies [11, 26].

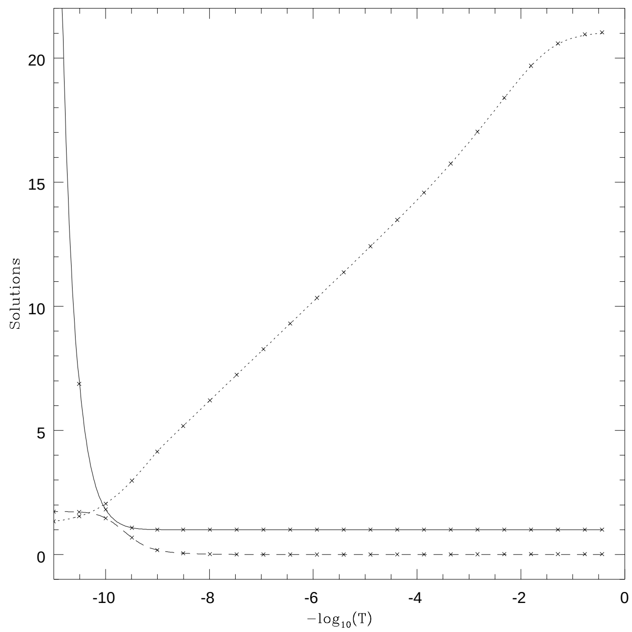

Figure 3. Numerical solutions for ξ (solid line), $\log_{10}[1+\alpha^{-2}]$ (dotted line) and φ' (dashed line) in an open ($k=-1$) scalar-tensor theory defined by $\alpha^2 = (\lambda^2/3)(1-\Phi)$ with $\lambda^2 = 0.9$. The results obtained from a backward and a forward time-integration are represented by lines and crosses, respectively.

Acknowledgments

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Appendix A: Numerical stability of the time-integration sense

We will here comment a series of computations which were performed in [21, 11] to test the numerical stability against the time-integration sense. In these computations, the ST cosmological equations were solved in the two senses (forward and backward in time) by using a sixth-order Runge-Kutta method which takes the temperature as variable. A standard particle content (baryons, electron-positrons, photons and neutrinos) of the universe is assumed, so that transition phases are solved without approximations.

Figure 3 shows a typical example of the numerical results found for the family of scalar-tensor theories considered in [21, 11]. The lines in this figure represent the solutions obtained for ξ (solid line), $\log_{10}(1+\alpha^{-2})$ (dotted line) and φ' (dashed line) from a backward time-integration. The final results of this integration were taken as boundary conditions to a reverse (forward in time) integration. The results obtained in this last case are represented by crosses in the same figure.

As we can see from Fig. 3, both integration senses give almost exactly the same results. This fact can be very well understood if one analyses the propagation of

numerical errors during the integration. To that end, we consider the simple case of a critically damped motion ($B_2 = \frac{3}{8}$), where the time evolution of φ is given by Eq. (41):

$$\varphi(p) = \varphi_0 \left[1 + \left(\frac{\varphi_0'}{\varphi_0} + \frac{3}{4}\right) p\right] e^{-\frac{3}{4}p} \tag{77}$$

Let now suppose that we integrate this equation backward in time (p varying from 0 today up to large negative values in the past) with numerical errors $\Delta\varphi_0 \ll \varphi_0$ and $\Delta\varphi_0' \ll \varphi_0'$ on the initial values. Obviously, since the functions φ and φ' increase in the past, their absolute errors will also increase. However the relevant quantity to estimate the accuracy of the integration is not the absolute error but the relative error which, at any p-time, is given by

$$\left(\frac{\Delta\varphi}{\varphi}\right)(p) = \frac{\Delta\varphi_0 + (\frac{3}{4}\Delta\varphi_0 + \Delta\varphi_0')p}{\varphi_0 + (\frac{3}{4}\varphi_0 + \varphi_0')p} \tag{78}$$

$$\left(\frac{\Delta\varphi'}{\varphi'}\right)(p) = \frac{-\Delta\varphi_0' + \frac{3}{4}(\frac{3}{4}\Delta\varphi_0 + \Delta\varphi_0')p}{-\varphi_0' + \frac{3}{4}(\frac{3}{4}\varphi_0 + \varphi_0')p} \tag{79}$$

In the limit where p goes to $-\infty$, the above expressions become

$$\left(\frac{\Delta\varphi}{\varphi}\right)_{-\infty} = \left(\frac{\Delta\varphi'}{\varphi'}\right)_{-\infty} = \frac{\frac{3}{4}\Delta\varphi_0 + \Delta\varphi_0'}{\frac{3}{4}\varphi_0 + \varphi_0'} \tag{80}$$

Since $\Delta\varphi_0 \ll \varphi_0$ and $\Delta\varphi_0' \ll \varphi_0'$, we then deduce that the relative errors remain very small in the past. This explains why the computation presented in Fig. 3 are stable in both integration senses.

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